

# A direct proof of the Stanley-Wilf Conjecture for permutations on multisets\*

Marie-Louise Bruner

Institut für Diskrete Mathematik und Geometrie

Technische Universität Wien

Vienna, Austria

marie.louise-bruner@tuwien.ac.at

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The concept of pattern avoidance respectively containment in permutations can be extended to permutations on multisets in a straightforward way. In this note we present a direct proof of the already known fact that the well-known Stanley-Wilf Conjecture, stating that the number of permutations avoiding a given pattern does not grow faster than exponentially, also holds for permutations on multisets.

## 1 Introduction

Since 1985 when Rodica Simion and Frank Schmidt published the first systematic study of *Restricted Permutations* [12] the area of pattern avoidance in permutations has become a rapidly growing field of enumerative combinatorics. A comprehensive overview of this area can be found in [3, 6]. We first state the central definition for permutations on a set:

**Definition 1.1.** A permutation  $p = p_1 p_2 \dots p_n$  of length  $n \geq k$  is said to contain another permutation  $q = q_1 q_2 \dots q_k$  as a pattern if we can find  $k$  entries  $p_{i_1}, p_{i_2}, \dots, p_{i_k}$  with  $i_1 < i_2 < \dots < i_k$  such that  $p_{i_a} < p_{i_b} \Leftrightarrow q_a < q_b$ . In other words,  $p$  **contains**  $q$  if we can find a subsequence of  $p$  that is order-isomorphic to  $q$ . If there is no such subsequence we say that  $p$  **avoids the pattern**  $q$ .

**Example 1.2.** The permutation  $p = 23718465$  (written in one-line representation respectively as a sequence of integers) contains the pattern 312, since the entries 714 (or several other examples) form a 312-pattern. See Figure 1 for a graphical representation with the help of permutation matrices, i.e. square binary matrices with exactly one entry equal to 1 - placed at  $(p(i), i)$  - per row and per column. As a matter of fact, this permutation contains all possible patterns of length three. This is not the

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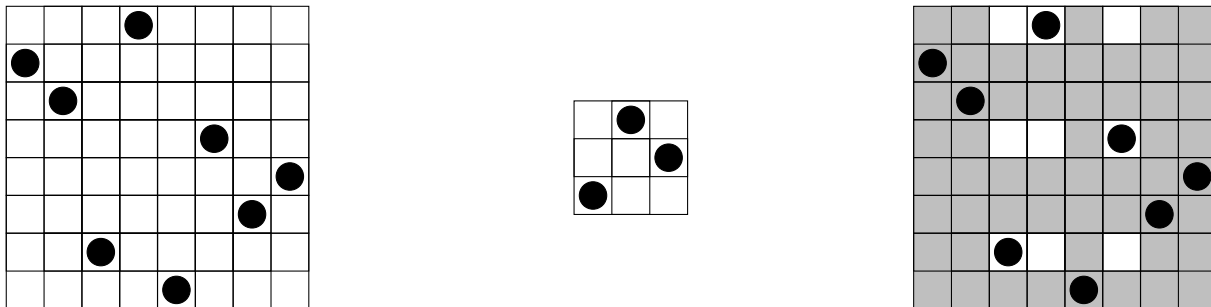


Figure 1: The permutation 23718465 (left) contains the pattern 312 (middle), as can be seen by deleting the rows and columns marked in grey (right).

case for patterns of length four:  $p$  contains the pattern 2134 as is shown by the entries 3146 but  $p$  avoids the pattern 4321 since it contains no decreasing subsequence of length four.

Considering *permutations on multisets* (or words on a certain alphabet) is a natural generalization of ordinary permutations. By a multiset we mean a “set”, where elements may occur more than once. We write  $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$  for the multiset in which the element  $i$  occurs  $m_i$ -times and call  $m_i$  the *multiplicity* of  $i$ . A multiset is called *regular* in case  $m_i = m$  for some integer  $m$  and we use the notation  $[n]_m$  for  $\{1^m, 2^m, \dots, n^m\}$  where  $[n] := [n]_1$ . The definition of pattern avoidance respectively containment introduced in Definition 1.1 can be extended to permutations on multisets in a straightforward way. For a certain pattern to be contained in a permutation, repetitions in the pattern have to be represented by repetitions in the permutation. For instance, the multiset-permutation 1214324 contains the patterns 122, 123 and 321 but avoids the pattern 211. enumerative questions for restricted permutations on multisets have been studied during the last decade, see e.g. the work of Heubach and Mansour [5] and Albert et. al. [1].

When attempting to compute the number  $S_n(q)$  of ordinary permutations of length  $n$  avoiding a certain pattern  $q$ , very satisfying results were obtained for patterns of length three:  $S_n(q) = c_n = \frac{1}{n+1} \binom{2n}{n}$ , the  $n$ -th Catalan number, for all six 3-patterns  $q$  (see e.g. [9, 12]). For longer patterns however, the situation turns out to be a lot more complicated: even for patterns of length four there is still one of the three *Wilf-equivalence* classes<sup>1</sup> for which no enumeration formula has yet been found. There is nevertheless a more general result on restricted permutations, stating that the number of  $n$ -permutations avoiding an arbitrary given pattern does not grow faster than exponentially:

**Theorem 1.3** (Stanley-Wilf Conjecture, 1990, proven 2004 in [7, 11]). *Let  $q$  be an arbitrary pattern. Then there exists a constant<sup>2</sup>  $c_q$  such that for all positive integers it holds that*

$$S_n(q) \leq c_q^n. \quad (1)$$

As Bona remarked in [3], this was quite an ambitious conjecture since it postulates that the number of  $q$ -avoiding  $n$ -permutations does not grow faster than exponentially whereas the total number of  $n$ -

<sup>1</sup>Two patterns  $q_1$  and  $q_2$  are *Wilf-equivalent* if  $S_n(q_1) = S_n(q_2)$  holds for all integers  $n$ .

<sup>2</sup>As a matter of fact, the constant  $c_q$  only depends on the length of the forbidden pattern  $q$ .

permutations grows super-exponentially, cf. the Stirling formula that states that  $n!$  is asymptotically equal to  $\sqrt{2\pi n} (n/e)^n$ .

A generalization of the Stanley-Wilf conjecture has been considered independently by Klazar and Marcus [8] and by Balogh, Bollobás and Morris [2]. Klazar and Marcus proved an exponential bound on the number of hypergraphs avoiding a fixed permutation, settling various conjectures of Klazar as well as a conjecture of Brändén and Mansour. Balogh, Bollobás and Morris went even further in their generalization and showed similar results for the growth of hereditary properties of partitions, ordered graphs and ordered hypergraphs. For details, please consider the original work. The results in [2, 8] being very general however required rather involved proofs. In both papers, a generalized version of the Füredi-Hajnal conjecture (Theorem 2.5) was first formulated and the proof of several intermediary results was necessary.

From the results in both these papers it follows that the Stanley-Wilf conjecture also holds for permutations on multisets since these can be represented with the help of bipartite graphs which are (very) special cases of hypergraphs. To the best of our knowledge it has however not yet been stated in the literature that the Stanley-Wilf conjecture also holds for permutations on multisets. We therefore wish to stress this point here, as pattern avoidance in permutations on multisets has attracted a great deal of interest in the past few years.

This present note is devoted to providing a simple and direct proof of the Stanley-Wilf Conjecture for permutations on multisets. The following proof uses the idea of Klazar [7] and does not require the employment of any other results or a further generalization of already known results.

## 2 A generalization of Stanley-Wilf to multisets

### 2.1 Multiset-Stanley-Wilf

Our goal is to prove the following generalization of Theorem 1.3 to permutations on regular multisets:

**Theorem 2.1** (Multiset-Stanley-Wilf). *Let  $q$  be an arbitrary permutation on an ordinary set and  $S_{n,m}(q)$  the number of permutations on the regular multiset  $[n]_m$  avoiding this pattern. Then there exists a constant  $e_q$  merely depending on the length of  $q$  so that the following holds for all positive integers  $n$  and  $m$ :*

$$S_{n,m}(q) \leq e_q^{n \cdot m}. \quad (2)$$

Multiset-Stanley-Wilf can easily be extended to permutations on an arbitrary multiset, see Remark 2.7.

*Remark 2.2.* Now, why would the upper bound suggested in (2) provide an equally strong result for permutations on multisets as Stanley-Wilf for ordinary permutations? How does  $e_q^{n \cdot m}$  compare to the total number of permutations on the regular multiset  $[n]_m$ ? For a regular multiset the total number of permutations is equal to

$$A_{n,m} := \binom{m \cdot n}{\underbrace{m, m, \dots, m}_{n\text{-times}}} = \frac{(mn)!}{(m!)^n}.$$

Let us set  $f(n, m) := \sqrt{2\pi mn} (n^m / (\sqrt{2\pi m} \cdot e))^n$ . Applying Stirling's formula then yields that  $f(n, m) \ll A_{n, m}$  when  $m$  is fixed and  $n \rightarrow \infty$ . Obviously  $f(n, m)$  grows a lot faster than  $e_q^{n \cdot m}$  for any constants  $e_q$  and  $m$ .

*Remark 2.3.* The condition that the avoided pattern  $q$  is a permutation on an ordinary set is crucial in Theorem 2.1. Indeed, the number  $S_{n, m}(q)$  may grow faster than exponentially if the pattern  $q$  is a multiset-permutation. For example, consider  $m$ -Stirling permutations, i.e. permutations on a regular multiset with multiplicity  $m$  avoiding the pattern 212. As stated in [10], the number of  $m$ -Stirling permutations with  $n$  distinct elements is equal to  $n!m^n \binom{n-1+1/m}{n}$  and thus grows super-exponentially.

For the proof of this multiset-version of Stanley-Wilf we follow the proof of the original result. Stanley-Wilf was not proven directly but via another conjecture formulated by Füredi and Hajnal.

## 2.2 The Füredi-Hajnal Conjecture

In [4] a conjecture concerning pattern avoidance in binary matrices was presented.

**Definition 2.4.** Let  $P$  and  $Q$  be matrices with entries in  $\{0, 1\}$  and let  $Q$  have the dimension  $m \times n$ . We say that the matrix  $P$  **contains**  $Q$  as a pattern, if there is a submatrix  $\tilde{Q}$  of  $P$ , so that  $\tilde{Q}_{i, j} = 1$  whenever  $Q_{i, j} = 1$  for  $i \leq m$  and  $j \leq n$ . If there is no such submatrix  $\tilde{Q}$ , we say that  $P$  **avoids**  $Q$ .

This means that  $P$  contains  $Q$  as a pattern, if, by deleting some rows and some columns, one can obtain a matrix  $\tilde{Q}$  with the same size as  $Q$  that has a 1-entry everywhere where  $Q$  has a 1-entry. Note that  $\tilde{Q}$  must not necessarily have its 0-entries in the same places as in  $Q$ :  $\tilde{Q}$  may have more 1-entries than  $Q$  but not less.

**Theorem 2.5** (Füredi-Hajnal Conjecture). Let  $Q$  be any permutation matrix. We define  $f(n, Q)$  as the maximal number of 1-entries that a  $Q$ -avoiding  $(n \times n)$ -matrix  $P$  may have. Then there exists a constant  $d_Q$  so that

$$f(n, Q) \leq d_Q \cdot n.$$

In the year 2000 Martin Klazar proved that the Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture [7]. Four years later the Füredi-Hajnal conjecture was proven by Adam Marcus and Gábor Tardos in [11], finally providing a simple and “gorgeous” [13] proof of the long-standing Stanley-Wilf conjecture. In their proof Marcus and Tardos showed that the constant  $c_q$  in Theorem 1.3 does not depend on the pattern  $q$  itself but merely on its length.

## 2.3 Klazar's proof

For the proof of Theorem 2.1, it will merely be necessary to show that Füredi-Hajnal implies the multiset-version of Stanley-Wilf. Let us therefore briefly recall the argument used by Martin Klazar in his proof.

In order to establish a connection between pattern avoidance in matrices and pattern avoidance in permutations Klazar takes an elegant detour via pattern avoidance in simple bipartite graphs and defines the following notion of pattern containment:

**Definition 2.6.** Let  $P([n], [n'])$  and  $Q([k], [k'])$  be simple bipartite graphs, where  $k \leq n$  and  $k' \leq n'$ . Then we say that  $P$  **contains  $Q$  as an ordered subgraph** if two order preserving injections  $f : [k] \rightarrow [n]$  and  $f' : [k'] \rightarrow [n']$  can be found so that if  $vv'$  is an edge of  $Q$ , then  $f(v)f'(v')$  is an edge of  $P$ .

Note again that - as in Definition 2.4 of pattern avoidance in binary matrices - the following holds: if  $f(v)f'(v')$  is an edge of  $P$ ,  $vv'$  does not necessarily have to be an edge of  $Q$ .

Clearly, every permutation can be identified with a simple bipartite graph in a unique way. For a permutation  $p$  on  $[n]$  the associated graph  $G_p$  is the bipartite graph with vertex set  $([n], [n])$  and where  $e = (i, j)$  is an edge iff  $p_i = j$  in  $p$  ( $i$  is an element of the first set of  $n$  elements,  $j$  is an element of the second one). Then the following is a direct consequence: If the permutation  $p$  contains a permutation  $q$  as a pattern, then  $G_p$  contains  $G_q$  as an ordered subgraph. Reversly, if  $p$  avoids  $q$ ,  $G_p$  will also avoid  $G_q$ . However, not every simple bipartite graph corresponds to a permutation (this is only the case if the vertex degree is equal to one for all vertices), thus  $S_n(q) \leq G_n(q)$ , where  $G_n(q)$  is the number of simple bipartite graphs on  $([n], [n])$  avoiding the graph  $G_q$  corresponding to a permutation  $q$ . In his proof, Klazar therefore aims at showing that for every permutation  $q$  there is a constant  $c_q$  so that  $G_n(q) \leq c_q^n$ .

Let  $P$  be a simple bipartite graph on  $([n], [n])$  that avoids  $G_q$ . Then the adjacency matrix  $A(P)$  of  $P$  avoids the adjacency matrix  $A(G_q)$  and Theorem 2.5 implies that  $A(P)$  can have at most  $d_q \cdot n = d_{A(G_q)} \cdot n$  entries equal to 1, respectively that  $G_p$  can have at most  $d_q \cdot n$  edges. By gradually contracting the graph  $P$  - reducing its size to half in every step without loosing the  $G_q$ -avoiding property - Klazar shows that this leaves at most an exponential number of possibilities for the graph  $P$ :  $G_n(q) \leq 15^{2d_q n}$ . Thus  $S_n(q) \leq c_q^n$  with  $c_q = 15^{2d_q}$ .

## 2.4 Using Klazar's idea for permutations on multisets

We shall use the same idea of contracting simple bipartite graphs in order to prove Theorem 2.1. It is straightforward to see how simple bipartite graphs can be used to represent permutations on multisets. Let  $p = p_1 p_2 \dots p_l$  be a permutation on the multiset  $[n]_m$ , i.e. a permutation of length  $l = n \cdot m$ . Then the associated bipartite graph  $G_p$  is a graph with vertex set  $([l], [n])$  and where  $e = (i, j)$ ,  $i \in [l]$  and  $j \in [n]$  is an edge iff  $p_i = j$ . Note that this bipartite graph is balanced iff  $m = 1$ , i.e. in the case of permutations on ordinary sets. See the left-hand side of Figure 2 for the representation of the multiset-permutations 1212 and 111.

*Proof of Theorem 2.1.* Let  $G_{n,m}(q)$  be the number of simple bipartite graphs on  $([n \cdot m], [n])$  avoiding the graph  $G_q$  corresponding to the ordinary permutation  $q$ . We shall show that  $G_{n,m}(q) \leq e_q^{n \cdot m}$ , implying that  $S_{n,m}(q) \leq e_q^{n \cdot m}$ . As in the case of ordinary permutations and balanced simple bipartite graphs it holds that  $S_{n,m}(q) \leq G_{n,m}(q)$  since not all simple bipartite graphs on  $([n \cdot m], [m])$  correspond to permutations on the multiset  $[n]_m$ .

Let  $G$  be a simple bipartite graph on  $([n \cdot m], [n])$  avoiding  $G_q$ . In a first step we shall contract the bipartite graph  $G$  to a balanced bipartite graph  $G'$  on the vertex set  $([n], [n])$ . For this purpose we merge  $m$  consecutive vertices of the  $[n \cdot m]$ -vertex set to a single one in the following way. If  $i \in [n]$  and  $j \in [n]$  are two vertices in  $G'$ , then let  $(i, j)$  be an edge if there is at least one edge between the set of vertices  $V_{m,i} := \{(m-1)i + 1, (m-1)i + 2, \dots, mi\}$  and  $j$  in  $G$ . For an example of such a contraction, see Figure 2.

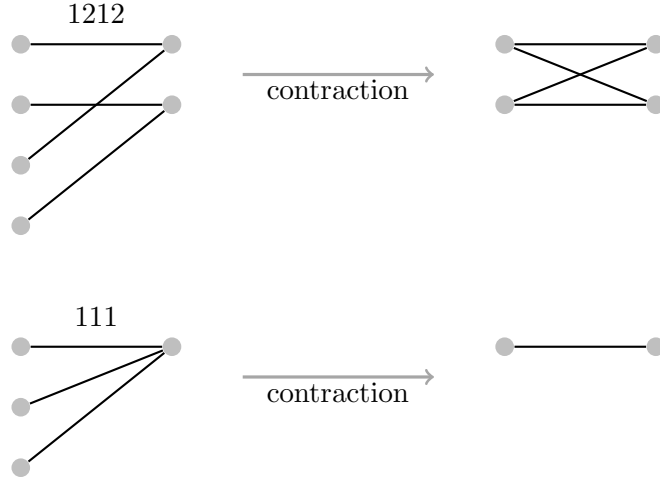


Figure 2: Why the proof of Theorem 2.1 does not work for multiset-patterns: The top-left graph avoids the bottom-left one but the contracted graph on the top-right no longer avoids the one on the bottom right.

The balanced bipartite graph  $G'$  inherits the  $G_q$ -avoiding property from  $G$ . Indeed, if it was possible to find a “copy” of  $G_q$  in  $G'$ , we could find “ancestors” of all the edges involved in this copy in  $G$ . Then these “ancestor-edges” would certainly form a copy of  $G_q$  in  $G$ .

Let us recapitulate: The resulting graph  $G'$  is a simple bipartite graph on  $([n], [n])$  avoiding  $G_q$ . If we consider the adjacency matrices  $A(G')$  respectively  $A(G_q)$ , i.e. the binary matrices for which  $a_{i,j} = 1$  iff  $(i, j)$  is an edge in the corresponding graph, it is clear that  $A(G')$  avoids  $A(G_q)$ . From Füredi-Hajnal (Theorem 2.5) it follows that  $A(G')$  can have at most  $d_{A(G_q)} \cdot n = d_q \cdot n$  entries equal to 1, implying that  $G'$  may have at most  $d_q \cdot n$  edges. As remarked earlier in 2.2, the constant  $d_q$  merely depends on the length of  $q$ .

How many different graphs  $G$  can lead to the same contracted graph  $G'$ ? For every edge  $(i, j)$  in  $G'$ , there are exactly  $2^m - 1$  different sets of edges between  $V_{m,i}$  and  $j$  leading to this specific edge. Indeed, every non-empty subset of  $\{(k, j) : k \in V_{m,i}\}$  leads to an edge between  $i$  and  $j$  in  $G'$ . Thus there are in total at most

$$(2^m - 1)^{d_q \cdot n}$$

possible graphs  $G$  contracting to the balanced graph  $G'$  and we obtain:

$$G_{n,m}(q) \leq (2^m - 1)^{d_q \cdot n} \cdot G_{n,1}(q).$$

Now that we have obtained a balanced bipartite graph  $G'$  avoiding  $G_q$ , we can continue in Klazar’s proof where it was shown that

$$G_{n,1}(q) = G_n(q) \leq 15^{2d_q n}.$$

This finally leads to

$$G_{n,m}(q) \leq ((2^m - 1) \cdot 15^2)^{d_q \cdot n},$$

proving that  $S_{n,m}(q) \leq e_q^{m \cdot n}$  where  $e_q = (2 \cdot 15^2)^{d_q}$  is a constant depending only on the length of the pattern  $q$ .  $\square$

## 2.5 Concluding remarks

*Remark 2.7.* Using the proof given above, it is clear why the Stanley-Wilf conjecture cannot hold for permutations on multisets avoiding some multiset-pattern. Indeed, it can easily be seen why the contraction does not work if the forbidden pattern itself is a permutation on a multiset. The crucial point in our proof is that the balanced graph resulting from the contraction of the original bipartite graph  $G$  still avoids the forbidden pattern-graph  $G_q$ . However, if the pattern itself must be contracted to a balanced bipartite graph, then the pattern-avoiding property is in general not passed on. For example, consider the (bipartite graph corresponding to the) permutation  $p = 1212$  represented in Figure 2. It clearly avoids the (graph corresponding to the) pattern  $q = 111$ , since no element occurs thrice in  $p$ . To the contrary, the contracted graph obviously contains the contracted pattern-graph since its edge-set is non-empty.

*Remark 2.8.* The above generalization of Stanley-Wilf to multisets was formulated for regular multisets - what happens if we consider arbitrary multisets instead of regular ones? The reasoning and remarks above are obviously also valid if we set  $m := \max \{m_i : i \in [n]\}$  for an arbitrary multiset. The contraction done in the first step of the proof of Theorem 2.1 must then be carried out in the following way: merge the first  $m_1$ -many vertices, then the following  $m_2$ -many vertices and so on until finally the last  $m_n$ -many vertices are merged to a single one.

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